Complexity Results for Enhanced Qualitative Probabilistic Networks

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Abstract

While quantitative probabilistic networks (QPNs) allow experts to state influences between nodes in the network as influence signs, rather than conditional probabilities, inference in these networks often leads to ambiguous results due to unresolved trade-offs in the network. Various enhancements have been proposed that incorporate a notion of strength of the influence, such as enhanced and rich enhanced operators. Although inference in standard (i.e., not enhanced) QPNs can be done in time polynomial to the length of the input, the computational complexity of inference in these enhanced networks has not been determined yet. In this paper, we introduce relaxation schemes to relate these enhancements to the more general case where continuous influence intervals are used. We show that inference in networks with continuous influence intervals is NP-hard, and remains NP-hard when the intervals are discretised and the interval $[-1, 1]$ is divided into blocks with length of $\frac{1}{4}$. We discuss membership of NP, and show how these general complexity results may be used to determine the complexity of specific enhancements to QPNs. Furthermore, this might give more insight in the particular properties of feasible and infeasible approaches to enhance QPNs.

1 Introduction

While probabilistic networks [9] are based on an intuitive notion of causality and uncertainty of knowledge, elicitating the required probabilistic information from experts can be a difficult task. Qualitative probabilistic networks [14], or QPNs, have been proposed as a qualitative abstraction of probabilistic networks to overcome this problem. These QPNs summarise the conditional

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probabilities between the variables in the network into a sign, which denotes the qualitative influence between these variables, i.e., the direction of the effect. In contrast to quantitative networks, where inference has been shown to be NP-hard [2], these networks have efficient (i.e., polynomial-time) inference algorithms.

Other uses of QPNs include their use as an intermediate step in the construction of a probabilistic network [12], a tool for verifying properties of such networks [7], and applications where the exact probability distribution is unknown or irrelevant [14].

Unfortunately, reasoning in a qualitative abstraction fails to give a conclusive result when influences with contrasting signs are combined. *Enhanced* QPNs have been proposed [10] in order to allow for more flexibility in determining the influences (e.g., *weakly* or *strongly* positive) and partially resolve conflicts when combining influences. Also, mixed networks [12] have been proposed to facilitate stepwise quantification by allowing both qualitative and quantitative influences to be modelled in the network.

Although inference in quantitative networks is NP-hard, and polynomial-time algorithms are known for inference in standard qualitative networks, the computational complexity of inference in enhanced networks has not been determined yet. In this paper we recall the definition of QPNs in Section 2, and we introduce a framework to relate various enhancements, such as enhanced, rich enhanced, and interval-based operators in Section 3. In Section 4 we show that inference in the general, interval-based case is NP-hard. In Section 5 we show that it remains NP-hard if we use discrete—rather than continuous—intervals. Furthermore, we argue that, although hardness proofs might be non-trivial to obtain, it is unlikely that there exist polynomial algorithms for less general variants of enhanced networks, such as the enhanced and rich enhanced operators suggested by Renooij and Van der Gaag [10]. Finally, we conclude our paper in Section 6.

2 Qualitative Probabilistic Networks

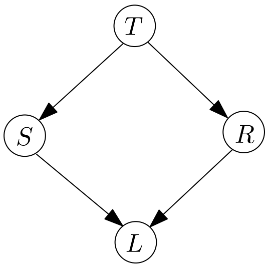
A qualitative probabilistic network $Q = (G, \Delta)$ is defined by associating a set Δ of qualitative influences and synergies [14] with a directed acyclic graph $G = (V, A)$. Such a network can be seen as an *abstraction* of a family of ‘traditional’ probabilistic networks, where the joint probability distribution of these networks respects the restrictions imposed by Δ. The influences and synergies in Δ are denoted by signs. For example, a *positive* influence of a node A on its successor B, denoted with $S^+(A, B)$, expresses that higher values for A make higher values for B more likely than lower values, regardless of influences of other nodes on B. In binary cases, with $a > \bar{a}$ and $b > \bar{b}$, this can be summarised as $\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geq 0$ for any value of x of other predecessors of B. *Negative* influences, denoted by S^-, and *zero* influences, denoted by S^0, are defined analogously.

If an influence is not positive, negative, or zero, it is *ambiguous*, denoted by $S^?$. This may be the case when the influence is non-monotone, e.g., $\Pr(b \mid ax_1) - \Pr(b \mid \bar{a}x_1) \geq 0$, but $\Pr(b \mid ax_2) - \Pr(b \mid \bar{a}x_2) \leq 0$.In addition, the sign '?' may occur during inference in cases where the actual influence is unknown (cannot be determined precisely). If this happens, the sign represents our lack of knowledge about the situation in the network, rather than the actual situation, and it is therefore desirable to generate as few of these signs as possible.

Influences can be direct (causal influence or influence along arcs) or *induced* (inter-causal influence or *product synergy*). In the latter case, the value of one node influences the probabilities of values of another (not directly connected) node, given a third node [4]. Furthermore, the notion of *additive synergy* is used to capture the joint effect of two variables on a third, rather than the effect of each variable separately. Both product and additive synergy are particularly useful when a QPN is used as an intermediate step in construction of a probabilistic network. They can model constraints in the probability distribution, without the need to specify the exact probabilities. Since they are not used for inference in QPNs, we will not discuss these synergies in this paper; the interested reader can refer to [4] or [14].

Example 1 (from [11]) *We consider a fragment of the Radiotherapy network which models the effect of radiotherapy on life expectancy (Figure 1). All variables are binary. Node L models a life-expectancy of at least six weeks, T models the presence or absence of radiotherapy, R models a reduction of the tumour, and S models the development of scar tissue. If a patient receives therapy, the tumour is likely to be reduced. On the other hand, scar tissue may develop. The associated conditional probabilities in the probabilistic network are summarised by '+' signs in the corresponding QPN. The tumour reduction increases the life expectancy of the patient; in contrast, the development of scar tissue decreases the life expectancy. These effects are summarised by the '+' respectively '−' signs.*

Various properties hold for these qualitative influences, namely *symmetry*, *transitivity*, *composition*, *associativity* and *distribution* properties, introduced in [14] and [10]. We define a *trail* t from A to B in a directed graph as a simple path from A to B in the underlying undirected graph, i.e., a list of arcs connecting A to B, regardless of the direction of the arcs. We define $\hat{S}^\delta(A, B, t)$ as the influence S^δ, with $\delta \in \{+, -, 0, ?\}$, from a node A on a node B along the trail t. The *symmetry* property indicates that influences along a trail are



$\Pr(t) =$	0.65		
$\Pr(s \mid t) =$	0.10	$\Pr(r \mid t) =$	0.85
$\Pr(s \mid \bar{t}) =$	0.01	$\Pr(r \mid \bar{t}) =$	0.35
$\Pr(l \mid sr) =$	0.70	$\Pr(l \mid s\bar{r}) =$	0.15
$\Pr(l \mid \bar{s}r) =$	0.75	$\Pr(l \mid \overline{sr}) =$	0.17

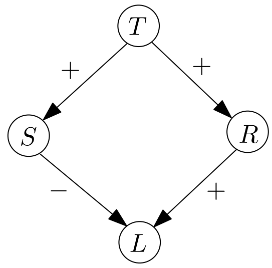


Fig. 1. The *Radiotherapy* network

independent of the direction of the arcs in the trail. In the example network, $\Pr(s \mid t) - \Pr(s \mid \bar{t}) = 0.10 - 0.01 \geq 0$. Likewise, using Bayes' rule we can calculate that $\Pr(t \mid s) - \Pr(t \mid \bar{s}) = 0.11 - 0.05 \geq 0$, so the influence of S on T has the same sign as the influence of T on S. This property holds in general.

The transitivity property defines the resulting sign of the chained effect between two variables along a trail using the $\otimes$-operator; the composition property defines the combined effect of a variable to another along multiple trails using the $\oplus$-operator. These properties are formalised as shown in Figure 2. The $\otimes$- and $\oplus$-operators that follow from the transitivity and composition properties are defined in Figure 3. In our example, we can infer from these properties that the positive effect of therapy on scar tissue also implies a positive effect in the opposite direction (symmetry); that the positive effects of therapy on tumour reduction, and of tumour reduction on life expectancy imply a positive effect of therapy on life expectancy along the trail $\{(T,R),(R,L)\}$ (transitivity); and that the opposite signs of the effects of therapy on life expectancy along the trails $\{(T,S),(S,L)\}$ and $\{(T,R),(R,L)\}$ imply an overall unknown effect (composition). The latter example illustrates the limited usefulness of QPNs when dealing with trade-offs in the network. The effect of therapy on life expectancy can be calculated to be positive, using the probabilities in the network, yet this effect is lost in the abstraction.

Using these properties, an efficient (polynomial time) inference algorithm can be constructed [3] that propagates observed node values to other neighbouring nodes, thus determining (As much as possible) the effect of the observation to the other nodes in the network. The basic idea of the algorithm, given in pseudo-code in Figure 4, is as follows. Initially, when no value has been observed, all *node signs* are 0. When entering the procedure, the observed node (say I) is instantiated with a '+' (for the observed value *true*) or a '−' (for the observed value *false*). Initially $trail = \emptyset$, $from = to = I$ and $msign =$ '+' or '−'. This node sign is propagated through the network to every active neighbour (i.e., every neighbour that is not independent of the observed node), combining the propagated node sign with already established node signs that stem from other trails from I. Observe from Figure 3 that a node sign can

Symmetry:	$\hat{S}^{\delta}(A,B,t_i) \in \Delta \Leftrightarrow \hat{S}^{\delta}(B,A,t_i^{-1}) \in \Delta$
Transitivity:	$\hat{S}^{\delta}(A,B,t_i) \wedge \hat{S}^{\delta'}(B,C,t_j) \Rightarrow \hat{S}^{\delta\otimes\delta'}(A,C,t_i \circ t_j)$
Composition:	$\hat{S}^{\delta}(A,B,t_i) \wedge S^{\delta'}(A,B,t_j) \Rightarrow S^{\delta\oplus\delta'}(A,B,t_i \parallel t_j)$
$\oplus$-Associativity:	$S^{(\delta\oplus\delta')\oplus\delta''} = S^{\delta\oplus(\delta'\oplus\delta'')}$
$\otimes$-Associativity:	$S^{(\delta\otimes\delta')\otimes\delta''} = S^{\delta\otimes(\delta'\otimes\delta'')}$
Distribution:	$S^{(\delta\oplus\delta')\otimes\delta''} = S^{(\delta\otimes\delta'')\oplus(\delta'\otimes\delta'')}$

Fig. 2. Properties of qualitative influences

$\otimes$	+	−	0	?
+	+	−	0	?
−	−	+	0	?
0	0	0	0	0
?	?	?	0	?

$\oplus$	+	−	0	?
+	+	?	+	?
−	?	−	−	?
0	+	−	0	?
?	?	?	?	?

Fig. 3. The $\otimes$- and $\oplus$-operator for combining signs

procedure PropagateSign(Trail *trail*, Node *from*, Node *to*, Sign *msign*):
to.sign $\leftarrow$ *to*.sign $\oplus$ *msign*;
trail $\leftarrow$ *trail* $\cup$ { *to* };
for each active neighbour V_i of *to*
do *lsign* $\leftarrow$ sign of influence between *to* and V_i;
    msign $\leftarrow$ *to*.sign $\otimes$ *lsign*;
    if $V_i \notin$ *trail* **and** V_i.sign $\neq V_i$.sign $\oplus$ *msign*
    then PropagateSign(*trail*, *to*, V_i, *msign*).

Fig. 4. The sign-propagation algorithm

change at most two times: from '0' to '+', '−', or '?', and then only to '?'. This algorithm visits each node at most two times, and therefore halts after a polynomial amount of time.

3 Enhanced QPNs

While these qualitative influences are useful to model influences between nodes in the network, a lot of information is lost in the abstraction. For example,

$\otimes_e$	$++^j$	$+^j$	$+_?$	0	$-_?$	$-^j$	$--^j$	$?$
$++^i$	$++^{i+j}$	$+^j$	$+_?$	0	$-_?$	$-^j$	$--^{i+j}$	$?$
$+^i$	$+^i$	$+^{i+j}$	$+^i$	0	$-^i$	$-^{i+j}$	$-^i$	$?$
$+_?$	$+_?$	$+^j$	$+_?$	0	$-_?$	$-^j$	$-_?$	$?$
0	0	0	0	0	0	$+$	$-$	0
$-_?$	$-_?$	$-^j$	$-_?$	0	$+_?$	$+^j$	$+_?$	$?$
$-^i$	$-^i$	$-^{i+j}$	$-^i$	0	$+^i$	$+^{i+j}$	$+^i$	$?$
$--^i$	$--^{i+j}$	$-^j$	$-_?$	0	$+_?$	$+^j$	$++^{i+j}$	$?$
$?$	$?$	$?$	$?$	$?$	$?$	$?$	$?$	$?$

Fig. 5. The enhanced $\otimes$-operator

trade-offs in the network can not be modelled. In the Radiotherapy network, the administration of radio therapy increases life expectancy because of the tumour reduction, but decreases life expectancy due to the development of scar tissue. The positive effect of tumour reduction, however, is much larger than the negative effect of the scar tissue as the conditional probability table in Figure 1 shows. Therefore, extensions to the QPN model, like the enhanced model in [10], have been suggested that preserve a larger amount of information in the abstraction than the traditional QPN model. For example, given a certain cut-off value α, an influence can be *strongly* positive ($\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geq \alpha$) or *weakly* negative ($-\alpha \leq \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leq 0$). The basic '+' and '−' signs are enhanced with signs for strong influences ('++' and '−−') and augmented with multiplication indices to handle complex dependencies on α as a result of transitive and compositional combinations. In addition, the signs '$+_?$' and '$-_?$' are used to denote positive or negative influences of unknown strength. This uncertainty may arise when combining effects during inference. For example, combining a strong positive with a weak negative effect leads to a positive effect of unknown strength. Multiplication indices occur when the influence of one node to another along a particular trail is calculated. For example, if $\Pr(b \mid a) - \Pr(b \mid \bar{a}) \geq \alpha$ and $\Pr(c \mid b) - \Pr(c \mid \bar{b}) \geq \alpha$, then $\Pr(c \mid a) - \Pr(c \mid \bar{a}) \geq \alpha^2$. Using this notion of strength, trade-offs in the network can be modelled by compositions of weak and strong opposite signs. The $\oplus$ and $\otimes$ operators associated with transitivity and composition properties in so-called *enhanced* QPNs are shown in Figures 5 and 6.

Furthermore, an *interval network* can be constructed [12], where each arc has an associated influence interval rather than a sign. Such an influence is denoted as $F^{[p,q]}(A, B)$, meaning that $\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \in [p, q]$, for every combination x of auxiliary parents of b. Note that, given this definition, $S^{+}(A, B) \iff F^{[0,1]}(A, B)$, and similar observations hold for S^{-}, S^{0} and $S^{?}$. We will denote the intervals $[-1, 0]$, $[0, 1]$, $[0, 0]$ and $[-1, 1]$ as *unit intervals*, being special cases that correspond to the traditional qualitative networks. The $\otimes$- and $\oplus$-operator, denoting transitivity and composition in interval networks are defined in Figure 7. Note that it is possible that a result of a combination

$\oplus_e$	$++^j$	$+^j$	$+_?$	0	$-_?$	$-^j$	$--^j$	$?$
$++^i$	$++^m$	$++^i$	$++^i$	$++^i$	$?$	$a)$	$?$	$?$
$+^i$	$++^j$	$+_?$	$+_?$	$+^i$	$?$	$?$	$d)$	$?$
$+_?$	$++^j$	$+_?$	$+_?$	$+_?$	$?$	$?$	$?$	$?$
0	$++^j$	$+^j$	$+_?$	0	$-_?$	$-^j$	$--^j$	$?$
$-_?$	$?$	$?$	$?$	$-_?$	$-_?$	$-_?$	$--^j$	$?$
$-^i$	$b)$	$?$	$?$	$-^i$	$-_?$	$-_?$	$--^j$	$?$
$--^i$	$?$	$c)$	$?$	$--^i$	$--^i$	$--^i$	$--^m$	$?$
$?$	$?$	$?$	$?$	$?$	$?$	$?$	$?$	$?$

where $m = min(i, j)$,

a) $+_?$, if $i \leq j$; ?, otherwise

b) $+_?$, if $j \leq i$; ?, otherwise

c) $-_?$, if $i \leq j$; ?, otherwise

d) $-_?$, if $j \leq i$; ?, otherwise

Fig. 6. The enhanced $\oplus$-operator

$\otimes_i$	$[r, s]$
$[p, q]$	$[\min X, \max X]$, where $X = \{p \cdot r, p \cdot s, q \cdot r, q \cdot s\}$

$\oplus_i$	$[r, s]$
$[p, q]$	$[p + r, q + s] \cap [-1, 1]$

Fig. 7. The $\otimes_i$- and $\oplus_i$-operators for interval multiplication and addition

of two trails leads to an empty set, for example when combining $[\frac{1}{2}, 1]$ with $[\frac{3}{4}, 1]$, which would denote that the total influence of a node on another node, along multiple trails, would be greater than one, which is impossible. Since the individual intervals might be estimated by experts, this situation is not unthinkable, especially in large networks. This property can be used to detect design errors in the network.

Note that the symmetry, associativity, and distribution property of qualitative networks no longer apply in these enhancements. For example, although a positive influence from a node A to B along the direction of the arc also has a positive influence in the opposite direction, the *strength* of this influence can not be determined. Also, the outcome of the combination of a strongly positive, weakly positive and weakly negative sign may be either unknown ('?') or positive, unknown strength ('$+_?$') depending on the evaluation order of the operators.

3.1 Relaxation schemes

If we take a closer look at the $\oplus_e$ and $\otimes_e$ operators defined in [10] and compare them with the interval operators $\oplus_i$ and $\otimes_i$, we can see that the interval results are sometimes somehow 'relaxed'. We see that symbols representing influences correspond to intervals, but after the application of any operation on these intervals, the result is extended to an interval that can be represented by one of the available symbols. For example, in the interval model we have $[\alpha, 1] \oplus_i [-1, 1] = [\alpha - 1, 1]$, but, while $[\alpha, 1]$ corresponds to $++$ in the enhanced model and $[-1, 1]$ corresponds to ?, $++ \oplus_e ? = ? \equiv [-1, 1]$. The lower limit $\alpha - 1$ is relaxed to -1, because the actually resulting interval $[\alpha - 1, 1]$ does not correspond to any symbol. Therefore, to connect the (enhanced) qualitative and interval models, we will introduce *relaxation schemes* that map the result of each operation to the minimal interval that can be represented by one of the available symbols.

Definition 2 *(Relaxation scheme) R_x will be defined as a relaxation scheme, denoted as $R_x([a, b]) = [c, d]$, if R_x maps the outcome $[a, b]$ of an $\oplus$ or $\otimes$ operation to an interval $[c, d]$, where $[a, b] \subseteq [c, d]$.*

In standard QPNs, the relaxation scheme (which we will denote R_I or the *unit scheme*) is defined as:

$$R_I(a, b) = \begin{cases} [0, 1] & \text{if } a \geq 0 \wedge b > 0 \\ [-1, 0] & \text{if } a < 0 \wedge b \leq 0 \\ [0, 0] & \text{if } a = b = 0 \\ [-1, 1] & \text{otherwise.} \end{cases}$$

Similarly, the $\oplus_e$ and $\otimes_e$ operators can be denoted with the following relaxation schemes, in which m equals $\min(i, j)$ and α is an arbitrary cut-off value.

$$R_{\otimes_e}(a, b) = \begin{cases} [-1, 1] & \text{if } a < 0 \wedge b > 0 \\ [a, b] & \text{otherwise.} \end{cases}$$

$$R_{\oplus_e}(a, b) = \begin{cases} [\alpha^m, 1] & \text{if } a = \alpha^i + \alpha^j \leq b \\ [-1, -\alpha^m] & \text{if } b = -(\alpha^i + \alpha^j) \geq a \\ [0, 1] & \text{if } a \leq b = \alpha^i + \alpha^j \\ [-1, 0] & \text{if } a = -(\alpha^i + \alpha^j) \leq b \\ [0, 1] & \text{if } a = (\alpha^i - \alpha^j) \text{ and } b \geq 0 \text{ and } i < j \\ [-1, 0] & \text{if } a = -(\alpha^i - \alpha^j) \text{ and } b \leq 0 \text{ and } i < j \\ [-1, 1] & \text{if } a \leq 0 \text{ and } b \geq 0 \\ [a, b] & \text{otherwise.} \end{cases}$$

The notion of a relaxation scheme allows us to relate various operators (like $\oplus_e$ and $\otimes_e$, but also the traditional (non-enhanced) operators and other enhancements like the *richly* enhanced $\oplus$ operator defined in [10]) in a uniform way. In the next section we will prove NP-hardness for the interval-based enhancements, and discuss the computational complexity of other enhancements in Section 5.

3.2 Problem definition

To decide on the complexity of inference of this general, interval-based enhancements of QPNs, a decision problem needs to be formulated. We state this problem, denoted as iPieqnetd [1], as follows.

iPieqnetd

Instance: Qualitative Probabilistic Network $Q = (G, \Delta)$ with an instantiation for $A \in V(G)$ and a node $B \in V \setminus \{A\}$.
Question: Is there an ordering on the combination of influences such that the computed influence of A on B is a strict subset of $[-1, 1]$?

To avoid problems associated with representing and manipulating real numbers, we assume that the probabilities in the network are fractions denoted by integer pairs rather than by reals. This has the advantage, that the length of the result of addition and multiplication of fractions is polynomial in the length of the original numbers [1].

4 Complexity of the problems

We will prove the hardness of the inference problem iPieqnetd by a transformation from 3sat. A 3sat instance is a logical formula in conjunctive normal form, that is, a conjunction over clauses, where each clause is a disjunction over literals (variables or their negation). In a 3sat instance, each clause has exactly three literals. The associated decision problem is, whether there exists an assignment to the variables that makes the formula true. The problem is known to be NP-complete [6].

We construct a network Q, with designated nodes I and Y, from a 3sat instance, consisting of ternary clauses C on Boolean variables U. We prove that, upon instantiation I to $[1, 1]$, an ordering on the combination of influences resulting in an influence on Y that is a true subset of $[-1, 1]$ exists, if and only if the corresponding 3sat instance is satisfiable. To improve readability, in the

[1] An acronym for *Interval-based Probabilistic Inference in Enhanced Qualitative Networks, Decision variant.*

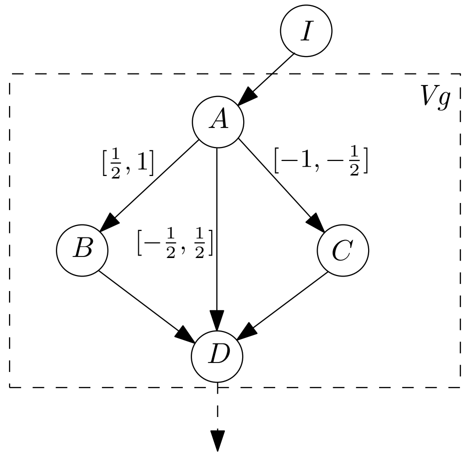


Fig. 8. “Variable gadget” Vg

remainder of this paper the $\oplus$- and $\otimes$-operators, when used without index, denote operators on intervals as defined in Figure 7.

In the network, the influence of a node A on a node B along the arc (A, B) is given as an interval; when the interval equals $[1, 1]$ (i.e., $\Pr(b \mid a) = 1$ and $\Pr(b \mid \bar{a}) = 0$) then the interval is omitted for readability.

As a running example, we will construct a network for the following 3SAT instance, introduced in [2]:

Example 3 *(*3SAT_{ex}*)*
$U = \{u_1, u_2, u_3, u_4\}$*, and*
$C = \{(u_1 \vee u_2 \vee u_3), (\neg u_1 \vee \neg u_2 \vee u_3), (u_2 \vee \neg u_3 \vee u_4)\}$.

This instance is satisfiable, for example with the truth assignment $u_1 = T$, $u_2 = F$, $u_3 = F$, and $u_4 = T$.

4.1 Construction for our proofs

For each variable u_i in the 3SAT instance, the network contains a “variable gadget” as shown in Figure 8. After the instantiation of node I with $[1, 1]$, the influence at node D equals $[\frac{1}{2}, 1] \oplus [-\frac{1}{2}, \frac{1}{2}] \oplus [-1, -\frac{1}{2}]$, which is either $[-1, \frac{1}{2}]$, $[-\frac{1}{2}, 1]$ or $[-1, 1]$, depending on the order of evaluation. We will use the non-associativity of the $\oplus$-operator in this network as a non-deterministic choice of assignment of truth values to variables. As we will see later, an evaluation order that leads to $[-1, 1]$ can be safely dismissed (it will act as a ‘falsum’ in all the clauses, making both u_i and $\neg u_i$ false), so we will concentrate on $[-\frac{1}{2}, 1]$ (which will be our T assignment) and $[-1, \frac{1}{2}]$ (F assignment) as the two possible choices.

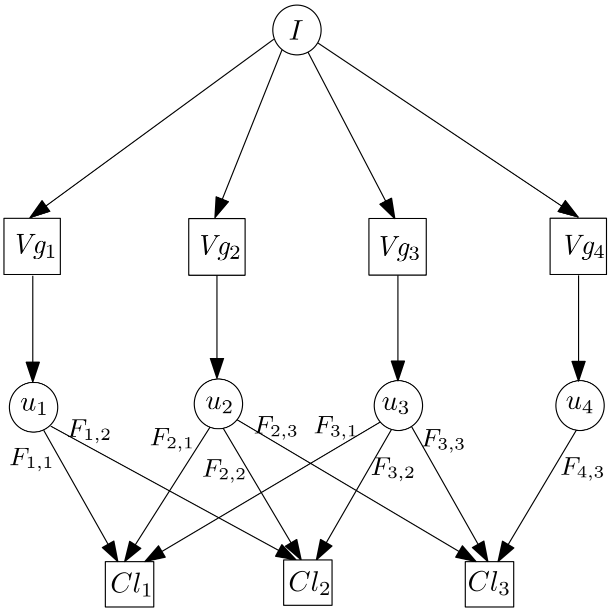


Fig. 9. The literal-clause construction

We construct subnetworks u_i from our 3SAT instance, each with a variable gadget Vg as input. Therefore, each variable can have a value of $[-1, \frac{1}{2}]$ or $[-\frac{1}{2}, 1]$ as influence, non-deterministically.

For each clause C_j, we add a clause-network Cl_j and connect the variable gadget of u_i to Cl_j if u_i occurs in C_j. The influence associated with this arc (u_i, Cl_j) is defined as $F^{[p,q]}(u_i, Cl_j)$, where $[p, q]$ equals $[-1, 0]$ if $\neg u_i$ is in C_j, and $[0, 1]$ if u_i is in C_j (Figure 9). Note that an $\otimes$-operation with $[-1, 0]$ will transform a value of $[-1, \frac{1}{2}]$ to $[-\frac{1}{2}, 1]$ and vice versa, and $[0, 1]$ will not change them. We can therefore regard an influence $F^{[-1,0]}$ as a negation of the truth assignment for that influence. Note, that the resulting influence $[-1, 1]$, not representing T or F, will stay the same in both cases. The clause-network is in Figure 10. The three 'incoming' variables in a clause (each of which has a value of either $[-1, \frac{1}{2}]$, $[-\frac{1}{2}, 1]$, or $[-1, 1]$) are multiplied with the arc influence $F_{i,j}$, and then combined with the instantiation node (with a value of $[1, 1]$), forming literal nodes w_i. Note that for a false literal, the value is $[-1, \frac{1}{2}] \oplus [1, 1] = [0, 1]$. For a true literal, the value is $[-\frac{1}{2}, 1] \oplus [1, 1] = [\frac{1}{2}, 1]$. Since the $[-1, 1]$ outcome of the variable gadget does not change by multiplication with the $F_{i,j}$ influence, the influence of the literal w_i would become $[-1, 1] \oplus [1, 1] = [0, 1]$, which is the same value as an F literal. In such an assignment to variable u_i, both the literal u_i and $\neg u_i$ will contribute the value F to any clause, and fail to satisfy it. If such an assignment can satisfy the 3SAT instance, the instance will also be satisfied with the assignment that gives u_i an arbitrary T or F value; hence the occurrence of these values can further be ignored.

The influences associated with these nodes w_i are multiplied by $[\frac{1}{2}, 1]$ and

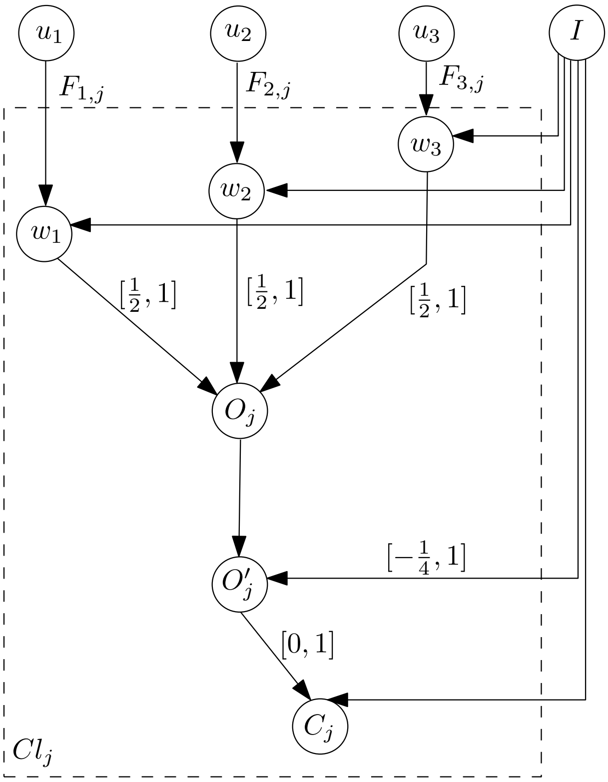


Fig. 10. The clause construction

added together in the clause intermediate result node O_j (Figure 10). At this point, O_j has a value of $[\frac{k}{4}, 1]$, where k equals the number of literals which are true in this clause. The consecutive addition of $[-\frac{1}{4}, 1]$, multiplication by $[0, 1]$ and addition of $[1, 1]$ mimics the function of a *logical or*-operator. Adding $[-\frac{1}{4}, 1]$ to $[\frac{k}{4}, 1]$ will lead to a positive interval if $kleq1$ and the interval $[-\frac{1}{4}, 1]$ if $k = 0$. As a result of the multiplication by $[0, 1]$, the positive interval will be 'stretched' to $[0, 1]$ for any $k \leq 1$ but remain $[-\frac{1}{4}, 1]$ if $k = 0$. Finally, adding $[1, 1]$ results in a value for the result node C_j of $[\frac{3}{4}, 1]$ if no literal in the clause was true, and $[1, 1]$ if one or more were true.

We then combine the separate clauses C_j into a variable D_n, by adding edges from each clause to D_n using intermediate variables D_1 to D_{n-1}. The use of these intermediate variables allows us later to generalise these results to more restricted cases. The interval of these edges is $[\frac{1}{2}, 1]$, leading to a value of $[1, 1]$ in D_n if and only if all clauses C_j have a value of $[1, 1]$ (see Figure 11). If one or more clause result nodes have a value of $[\frac{3}{4}, 1]$ (i.e., the clause is not satisfied by the variable instantiation), the influence interval in D_n has a value between $[\frac{3}{4}, 1]$ and $[\frac{2^{k+1}-1}{2^{k+1}}, 1]$ (where k is the number of clauses). Finally, we construct the output node Y by consecutively adding the influence in D_n to $[-1, 1]$ and $[-1 + \frac{1}{2^{k+1}}, 1]$ (Figure 12). This would result in a true subset of $[-1, 1]$ (namely, $[-1 + \frac{1}{2^{k+1}}, 1]$) if and only if the influence in D_n is equal to

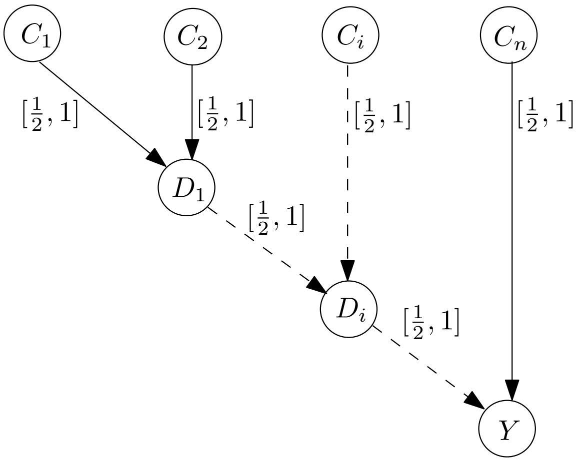


Fig. 11. Connecting the clauses

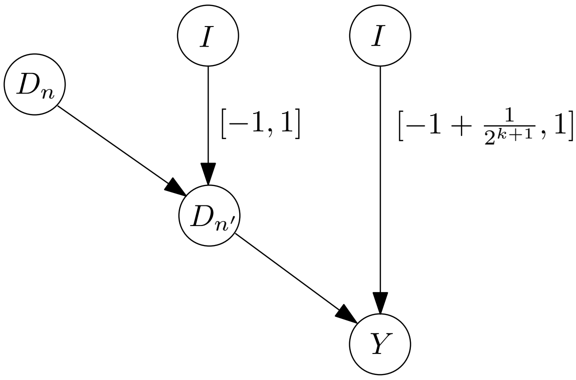


Fig. 12. Constructing the output node Y

$[1, 1]$, i.e, if all clauses are satisfied. If one or more clauses are not satisfied, then the influence in D_n will be at most $[\frac{2^{k+1}-1}{2^{k+1}}, 1]$ and the consecutive additions will ensure an influence in Y of $[-1, 1]$.

4.2 NP-hardness proof

Using the construct presented in the previous section, the computational complexity of the iPieqnetd can be established as follows.

Theorem 4 *The* iPieqnetd *problem is NP-hard.*

Proof. To prove NP-hardness, we construct a transformation from the 3sat problem. Let (U, C) be an instance of this problem, and let $Q_{(U,C)}$ be the interval-based qualitative probabilistic network constructed from this instance, as described in the previous section. When the node $I \in Q$ is instantiated with $[1, 1]$, then I has an influence of $[1, 1]$ on D_n (and therefore an influence on Y

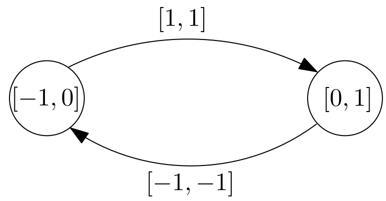


Fig. 13. Continuous node interval change

which is a true subset of $[-1,1]$) if and only if all nodes C_j have a value of $[1,1]$, i.e., there exists an ordering of the operators in the “variable-gadget” such that at least one literal in C is true. We conclude that (U,C) has a solution with at least one true literal in each clause, if and only if the iPieqnetd problem has a solution for network $Q_{(U,C)}$, instantiation $I=[1,1]$ and output node Y. Since $Q_{(U,C)}$ can be computed from (U,C) in polynomial time, we have a polynomial-time transformation from 3sat to iPieqnetd, which proves NP-hardness of iPieqnetd. □

4.3 On the possible membership of NP

Although iPieqnetd has been shown to be NP-hard, membership of NP (and, as a consequence, NP-completeness) is not trivial to prove. To prove membership of NP, one has to prove that if the instance is solvable, then there exists a certificate that can be used to verify this claim in polynomial time. A trivial certificate could be a formula, using the $\oplus$- and $\otimes$-operators, influences, and parentheses, describing how the influence of the a certain node can be calculated from the instantiated node and the characteristics of the network. Unfortunately, such a certificate can grow exponentially large, and verifying a claim using this certificate would take time, exponential in the size of the network. While the properties of the traditional (i.e., non-enhanced) $\otimes$- and $\oplus$-operators ensure that a node sign van change at most two times in an inference algorithm, this no longer holds for the interval operators. From the definition of the $\oplus_i$ operator in Figure 7, we can see that a node interval can change with every update and does not need to converge (see Figure 13 for an example). Thus, the number of interval changes can be as large as the (possibly exponential) number of trails between the instantiation node and the target node.

5 Operator variants

We now discuss whether the results from the previous section can be generalised to other operator variants. In order to be able to represent every possible 3SAT instance, a relaxation scheme must be able to generate a variable gadget, and retain enough information to discriminate between the cases where zero, or more literals in each clause are true. Furthermore, the relaxation scheme must be able to represent the instantiations $[1, 1]$ (or $\top$) and $[-1, -1]$ (or $\perp$), and the uninstantiated case $[0, 0]$. With a relaxation scheme that effectively divides the interval $[-1, 1]$ in discrete blocks with size of a multiple of $\frac{1}{4}$, (such as $R_{\frac{1}{4}}(a, b) = [\frac{\lfloor 4a \rfloor}{4}, \frac{\lceil 4b \rceil}{4}]$) the proof construction is essentially the same as in the general case discussed in section 3. This relaxation scheme does not have any effect on the intervals we used in the variable gadget and the clause construction of $Q_{(U,C)}$. The network constructed in the NP-hardness proof of the general case used only intervals (a, b) for which $R_{\frac{1}{4}}(a, b) = (a, b)$. Furthermore, when connecting the clauses, the possible influences in Y are relaxed to $[0, 1]$, $[\frac{1}{4}, 1]$, $[\frac{1}{2}, 1]$, $[\frac{3}{4}, 1]$, and $[1, 1]$, so we can construct Y' by consecutively adding the interval in Y to $[-1, 1]$ and $[-\frac{3}{4}, 1]$. Thus, the problem—which we will denote as RELAXED-PIEQNETD—remains NP-hard for relaxation scheme $R_{\frac{1}{4}}$.

The non-associativity of the $\oplus_e$–operator defined in [10] suggest hardness of the inference problem as well. Although $\oplus_e$ is not associative, it cannot produce results that can be regarded as opposites. For example, the expression $(+ + \oplus_e + \oplus_e -)$ can lead to a positive influence of unknown strength ('$+_?$') when evaluated as $((++\oplus_e+)\oplus_e-)$ or an unknown influence ('?') when evaluated as $(++\oplus_e(+\oplus_e-))$, but never to a negative influence. A transformation from a 3SAT variant might not succeed because of this reason. However, it might be possible to construct a transformation from RELAXED-PIEQNETD, which is subject of ongoing research.

6 Conclusion

In this paper, we addressed the computational complexity of inference in enhanced Qualitative Probabilistic Networks. As a first step, we have "embedded" both standard and enhanced QPNs in the interval-model using relaxation schemes, and we showed that inference in this general interval-model is NP-hard and remains NP-hard for relaxation scheme $R_{\frac{1}{4}}(a, b)$. In general, inference in interval-based networks is as hard as inference in probabilistic networks. Propagation algorithms for the interval-based networks may need exponential time, for example when a network has a high treewidth and there are many trails between observation node and output node. Nevertheless, efficient algorithms may be developed for networks that have a bounded treewidth, like the algorithm for probabilistic networks discussed in[8].

We believe that the hardness of inference is due to the fact that reasoning in QPNs is under-defined: The outcome of the inference process depends on choices during evaluation. Further research needs to be conducted in order to determine where exactly the *NP/P* border lies, in other words: which enhancements to the standard qualitative model allow for polynomial-time inference, and which enhancements lead to intractable results. Despite of the unfavorable complexity of inference, enhanced and interval-based qalitative models are useful as an intermediate step in the construction of a probabilistic network[12]. Nevertheless, a better definition of transitive and compositional combinations of qualitative influences in which the outcome is independent of the order of the influence propagation might reduce the computational complexity of inference, and facilitate the use of qualitative models to design, validate, analyse, and simulate probabilistic networks.

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