Letter to the Editor

Bayesian Intractability is not an Ailment that Approximation Can Cure

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The popularity of probabilistic (Bayesian) modeling is growing in cognitive science as evidenced by an increase in the number of articles, conference papers, symposia and workshops on the topic.[1] The popularity of the Bayesian modeling framework can be understood as a natural outflow of its success in producing models that describe and predict a wide variety of cognitive phenomena in domains ranging from vision (Yuille & Kersten, 2006), categorization (Anderson, 1990; Griffiths, Sanborn, Canini, & Navarro, 2008), decision-making (Sloman & Hagmayer, 2006) and language learning (Chater & Manning, 2006; Frank, Goodman, & Tenenbaum, 2009) to motor control (Körding & Wolpert, 2004, 2006) and theory of mind (Baker, Saxe, & Tenenbaum, 2009). Notwithstanding the empirical success of the Bayesian framework, models formulated within this framework are known to often face the theoretical obstacle of computational intractability. Formally, this means that computations that are postulated by many Bayesian models of cognition fall in the general class of so-called NP-hard problems. Informally, this means that the computations postulated by such models are too resource demanding to be plausibly performed by our resource-bounded minds/brains in a realistic amount of time.[2]

Bayesian modelers seem to be aware that their models often face the theoretical charge of intractability. Yet, we observe that they seem eager to downplay the real challenge posed by 'intractability' and are quick to claim that—despite the intractability of exact algorithms—Bayesian computations can be efficiently approximated using inexact algorithms (see e.g. Chater et al. (2006); Sanborn et al. (in press)). Although we agree that human minds/brains likely implement all kinds of inexact or quick-and-dirty algorithms, in this letter we wish to draw attention to the fact that this assumption alone is insufficient for Bayesian modelers to guarantee tractability of their models. The reason is, simply put, that intractable Bayesian

[1]For example, a search on keyword "Bayesian" in the proceedings of the 2000, 2004, and 2010 annual meetings of the Cognitive Science Society for papers and posters either using or discussing Bayesian methods shows an increase from one symposium and 5 papers in 2000 and 12 papers in 2004 to 100+ papers and posters in 2010.

[2]The purpose of this letter is not to argue for why NP-hard problems are indeed generally intractable—as this has been done extensively elsewhere (van Rooij, 2008)—nor to argue that intractability is really at stake for Bayesian models—as this is generally acknowledged by critics (Gigerenzer & Todd, 1999; Gigerenzer, 2008) and proponents (Chater, Tenenbaum, & Yuille, 2006; Sanborn, Griffiths, & Navarro, in press) of Bayesian modeling alike.

computations are not generally tractably approximable. This is not to say, of course, that cognitive algorithms do not approximate Bayesian computations, but rather to claim that approximation by itself cannot guarantee tractability.

With this letter, we wish to communicate two important points with the cognitive science community: First, current claims of tractable approximability of intractable (Bayesian) models in the cognitive science are mathematically unfounded and often provably unjustified. Second, there are a variety of complexity-theoretic tools available that Bayesian modelers can use to assess the (in)tractability of their models in a mathematically sound way.

To make our points, we will use a widely adopted—see e.g. Yuille and Kersten (2006); Baker et al. (2009); Chater and Manning (2006)—subcomputation of cognitive Bayesian models as an illustrative example: probabilistic abduction, a.k.a. *most probable explanation* (MPE). In brief, this computation is defined by the following input-output mapping:

> Most Probable Explanation (MPE)
>
> *Input:* A set of hypotheses H, a set of observations E, and a knowledge structure K encoding the probabilistic dependencies between observations, hypotheses, and possibly intermediate variables (e.g., K could be a Bayesian network).
>
> *Output:* A truth assignment for each hypothesis in H with the largest possible conditional probability over all such assignments (more formally, $\text{argmax}_{T(H)} Pr(T(H)|E, K)$ where T is a function $T : H \rightarrow \{true, false\}$).

In the computer science literature the computational complexity of MPE has been extensively studied. Not only is it known that computing MPE is NP-hard (Shimony, 1994), but it is also known that 'approximating' MPE—in the sense of computing a truth assignment that has close to maximal probability—is NP-hard (Abdelbar & Hedetniemi, 1998). An even more sobering result is that it has been proven NP-hard to compute a truth assignment with a conditional probability of at least q for any value $0 < q < 1$ (Shimony, 1994). Computational complexity results such as these show that claims in the cognitive science literature about the tractable approximability of intractable Bayesian computations are not generally warranted.

We realize that our message may seem counterintuitive from the perspective of the algorithmic-level modeler who implements probabilistic or randomized algorithms for approximating Bayesian computations and who may find that such algorithms may run quite fast and perform quite well. The paradox can be understood as a mismatch between the generality of the (intractable) computational-level Bayesian models and the (tractable) algorithms implemented for 'approximating' the postulated input-output mappings. The algorithms will run fast and perform well only for a proper subset of input domains, viz., those domains for which the computation (exact or approximate) is tractable.

A general methodology for identifying restricted domains of inputs for which otherwise intractable computations are tractable is available in computer science (Downey & Fellows, 1999; van Rooij & Wareham, 2008). The methodology consists of assuming constraints on the input domain and using proof techniques from parameterized complexity theory to show whether or not those constraints render the postulated computations tractable. For instance, it is known that if the knowledge structure underlying MPE is a Bayesian network with a special, constrained form of connectivity (known as 'bounded tree width'; Bodlaender (1997)) then computing MPE is tractable (Kwisthout, 2009, 2010; Sy, 1992). This means that for all cognitive domains in which such special connectivity can be assumed (e.g., psychologically and/or ecologically motivated) then the Bayesian computation MPE is no longer intractable, and approximation algorithms can tractably approximate that computation. Importantly, the tractability is achieved not by giving up on the 'exactness' of the postulated computations, but by explicating that the modeled process operate on restricted domains.

In closing, we wish to emphasize that our purpose is certainly not to downgrade Bayesian models of cognition or to argue that the challenge of making such models tractable cannot be met. On the contrary, we see great potential for Bayesian models to help advance the field of cognitive science and we hope to contribute to this by sharing our observations and pointing to a mathematically sound methodology for making Bayesian models tractable. Given that Bayesian modeling is now an accepted framework for approaching cognition, we think the time is ripe to start thinking seriously about the scalability of these models to real-world scenarios. In doing so, we believe that the Bayesian modeling community can (and should) display the same mathematical and scientific rigor as they have in probabilistic modeling and statistical testing. Making mathematically unfounded claims of efficient approximability of intractable Bayesian computations is *not* the way to move forward.

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